An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 41 (2001) 1589–1602
www.elsevier.nl/locate/camwa

An Approximate Approach for Ranking Fuzzy Numbers Based on Left and Right Dominance

LIANG-HSUAN CHEN AND HAI-WEN LU
Department of Industrial Management Science
National Cheng Kung University, Tainan, Taiwan, R.O.C.

(Received December 1999; accepted January 2000)

Abstract—This study presents an approximate approach for ranking fuzzy numbers based on the left and right dominance. The proposed approach only requires a few left and right spreads at some α-levels of fuzzy numbers to determine the respective dominance of one fuzzy number over the other. The total dominance is then determined by combining the left and right dominance based on a decision maker's optimistic perspectives. Such a dominance is useful in ranking the fuzzy numbers when membership functions cannot be acquired. The approach proposed herein is relatively simple in terms of computational efforts and is efficient when ranking a large quantity of fuzzy numbers. By using a few left and right spreads, two groups of examples demonstrate the accuracy and applicability of the proposed approach.

Keywords—Ranking fuzzy numbers, Dominance, Decision making.

1. INTRODUCTION

Decision makers are normally faced with the lack of precise information to assess a set of alternatives in an uncertain environment. Imprecise evaluations may be attributed to

(1) unquantifiable information,
(2) incomplete information,
(3) nonobtainable information, and
(4) partial ignorance [1].

To resolve this problem, the fuzzy set theory pioneered by Zadeh [2] has been extensively used. Fuzzy numbers or fuzzy subsets of the real line $\Re$ are applied to represent the imprecise numerical measurements of different alternatives. Therefore, comparing the different alternatives is actually comparing the resulting fuzzy numbers.

More than twenty fuzzy ranking indices have been proposed since 1976 [1,3]. Various techniques are applied to compare the fuzzy numbers. Some investigations [3–15] defined a ranking function to map a fuzzy number to a real number and, then, used natural orderings. Other investigators [16–25] defined a comparison function that maps two fuzzy numbers to a real number when determining the degree to which one dominates the other. Most approaches are based on the possibility concept and/or the probability measure of fuzzy events concept [3]. Several

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PII: S0898-1221(01)00124-9

authors [3,9,26,27] have reviewed and compared some of them using the same set of examples, as provided by Bortolan and Degani [28]. Chen and Hwang [1] thoroughly reviewed the existing approaches, pointing out some illogical conditions that arise among them.

Some of the existing approaches are difficult to understand and have suffered from different plights, e.g., the lack of discrimination, producing counterintuitive orderings, and ultimately resulting in inconsistent orderings if a new fuzzy number is added; high complexity and cumbersome computational efforts are also characteristic [29,30]. Nearly all approaches should acquire membership functions of fuzzy numbers before the ranking is performed; however, this may be infeasible in real applications. Furthermore, accuracy and efficiency should be of priority concern in the ranking process if ranking a large amount of fuzzy numbers.

In light of the above discussion, this study presents an approximate ranking approach based on the left and right dominance, which follows the concept of area measurement. Many ranking methods have already been developed to rank fuzzy numbers based on area measurement. Some of those methods compute the Hamming distance measurements between each fuzzy number and fuzzy maximum (or minimum) as the ranking basis, such as in the investigations of Yager [14], Kerre [8], Nakamura [22], and Kolodziejczyk [10]. However, these methods are illogical owing to the neglect of the fuzzy number's relative locations on the X-axis [1]. Tseng and Klein [23] proposed a ranking algorithm based on the difference concept. A preference relation is developed for the ranking process using the dominance and indifference, which adheres to the concept of difference between two fuzzy numbers. Yager [15] proposed a ranking index, F_3, by measuring the area from the membership axis to the average of the left and right membership functions. This index has consistent comparisons in Bortolan and Degani's examples [27,28]. Liou and Wang [11] ranked fuzzy numbers using the total integral values according to a decision maker's attitude of risk. Using their method, a neutral decision maker, who specifies the value (γ) of the index of optimum to be 0.5, will obtain the total integral value equivalent to F_3. Fortemps and Roubens [27] recently proposed a ranking method based on the concepts of area compensation, which corresponds to Yager's F_3 and the total integral value. Fortemps and Roubens's method produces equivalent outcomes as Yager's F_3 and Liou and Wang's total integral value with $\gamma = 0.5$, if the fuzzy numbers are normal and convex.

The above investigations require membership functions when computing the area compensation and the total integral value. Chen and Klein [29,30] employed several existing concepts to develop a ranking method based on area measurement without membership functions. Two crisp maximizing and minimizing barriers, proposed by Choobineh and Li [26], are first defined to construct a referential rectangle. The ranking index is then determined by the difference between the fuzzy number and the referential rectangle. However, the index is affected by the choice of the two barriers.

For circumventing the above-mentioned problems, the proposed approximate approach only uses α-cuts and performs simple arithmetic operations for the ranking purpose. Initially, the left (right) dominance is determined by summing the difference of the left (right) spreads at each α-level to denote the degree to which one fuzzy number dominates the other at the left(right)-hand side. According to our results, the left (right) dominance approximates the area difference of two fuzzy numbers from the membership axis to the left (right) membership function, when the number of α-cuts approaches the infinity. Moreover, to reflect the decision maker's optimistic or pessimistic perspectives, a convex combination of the left and right dominance using an index of optimism is employed to rank the fuzzy numbers [9]. The proposed approach corresponds to the F_3 index, the total integral value with $\gamma = 0.5$, and Fortemps and Roubens' area compensation approximately, when the index of optimism is assigned 0.5. An example is used to compare two fuzzy numbers, while considering different values of the index of optimism. Particularly, the proposed approach can also be applied to rank the combination case of some fuzzy numbers and crisp numbers and the case of discrete fuzzy numbers. Also described herein are some properties which are useful in ranking a large quantity of fuzzy numbers.

Comparing the proposed approximate approach with the existing methods using both Bortolan and Degani's examples [28] and Tseng and Klein's examples [23] reveals that the former is more simple, efficient, and consistent. The rest of this paper is organized as follows. Section 2 introduces the ranking approach. Section 3 describes some useful properties. Next, Section 4 presents some comparative examples which demonstrate the accuracy and efficiency of the proposed approach over the existing methods. Concluding remarks are finally made in Section 5.

2. THE RANKING METHOD

A real fuzzy number can be defined as a fuzzy subset of the real line $\Re$, which is convex and normal [31,32]. That is, for a fuzzy number A of $\Re$ defined by the membership function $\mu_A(x)$, $x \in \Re$, the following relations exist:

$$\max_x \mu_A(x) = 1, \tag{1a}$$

$$\mu_A[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_A(x_1), \mu_A(x_2)], \tag{1b}$$

where $x_1, x_2 \in \Re$, $\forall \lambda \in [0,1]$. A fuzzy number A with the membership function $\mu_A(x)$, $x \in \Re$, can be defined as [33]

$$\mu_A(x) = \begin{cases} \mu_A^{\mathrm{L}}(x), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ \mu_A^{\mathrm{R}}(x), & c \leq x \leq d, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $\mu_A^{\mathrm{L}}(x)$ is the left membership function that is an increasing function and $\mu_A^{\mathrm{L}} : [a,b] \to [0,1]$. Meanwhile, $\mu_A^{\mathrm{R}}(x)$ is the right membership function that is a decreasing function and $\mu_A^{\mathrm{R}} : [c,d] \to [0,1]$. In addition, a trapezoidal fuzzy number is denoted by $[a,b,c,d]$. In particular, $[a,b,c,d]$ can also signify a triangular fuzzy number if $b = c$. Assume that every fuzzy number is bounded; i.e., $-\infty < a, d < \infty$.

For a fuzzy number A, the α-cuts (level sets) $A_\alpha = \{x \in \Re \mid \mu_A(x) \geq \alpha\}$, $\alpha \in [0,1]$, are convex subsets of $\Re$. The lower and upper limits of the k^{th} α-cut for the fuzzy number A_i are defined as

$$l_{i,k} = \inf_{x \in \Re} \{x \mid \mu_A(x) \geq \alpha_k\}, \tag{3a}$$

$$r_{i,k} = \sup_{x \in \Re} \{x \mid \mu_A(x) \geq \alpha_k\}, \tag{3b}$$

respectively, where $l_{i,k}$ and $r_{i,k}$ are left and right spreads, respectively [29]. While comparing two fuzzy numbers A_i and A_j, Figure 1 illustrates their corresponding left and right spreads at the α_k level.

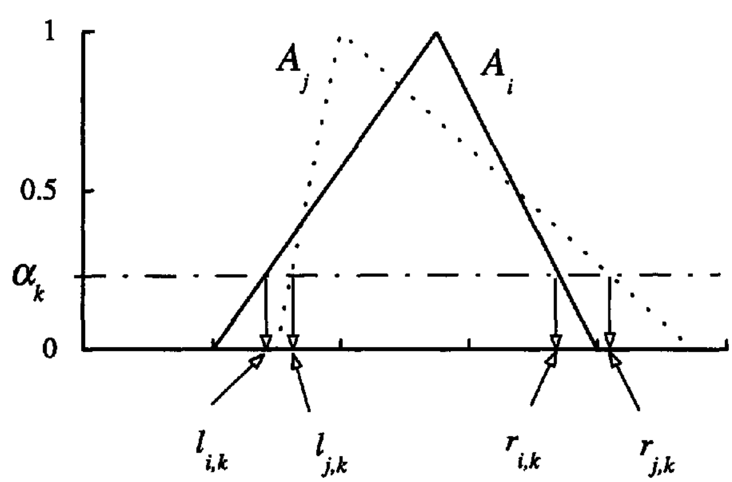

Figure 1. The left and right spreads of fuzzy numbers A_i and A_j.

The left (right) dominance $D^{\mathrm{L}}_{i,j}(D^{\mathrm{R}}_{i,j})$ of A_i over A_j is defined as the average difference of the left (right) spreads at some α-levels. They are formulated as

$$D^{\mathrm{L}}_{i,j} = \frac{1}{n+1}\sum_{k=0}^{n}(l_{i,k} - l_{j,k}) \tag{4a}$$

and

$$D^{\mathrm{R}}_{i,j} = \frac{1}{n+1}\sum_{k=0}^{n}(r_{i,k} - r_{j,k}), \tag{4b}$$

where $n+1$ α-cuts are used to calculate the dominance. Let α_k denote the k^{th} α-level, and $\alpha_k = k/n$, $k \in \{0, 1, \dots, n\}$. Therefore, the distance between each two adjacent α-levels is equal; i.e., $\alpha_k - \alpha_{k-1} = 1/n$, $k \geq 1$. As $n \to \infty$, $D^{\mathrm{L}}_{i,j}(D^{\mathrm{R}}_{i,j})$ approximates the area difference of A_i over A_j according to the membership axis to the left (right) membership function. The left and right spreads are equivalent to each other when the α-level passes the peak of a triangular fuzzy number A_i; i.e., $l_{i,n} = r_{i,n}$. In particular, the total dominance of A_i over A_j with the index of optimism $\beta \in [0,1]$ can be defined as the convex combination of $D^{\mathrm{L}}_{i,j}$ and $D^{\mathrm{R}}_{i,j}$ by

$$\begin{aligned} D_{i,j}(\beta) &= \beta D^{\mathrm{R}}_{i,j} + (1-\beta)D^{\mathrm{L}}_{i,j} \\ &= \beta\left[\frac{1}{n+1}\sum_{k=0}^{n}(r_{i,k} - r_{j,k})\right] + (1-\beta)\left[\frac{1}{n+1}\sum_{k=0}^{n}(l_{i,k} - l_{j,k})\right] \\ &= \frac{1}{n+1}\left\{\left[\beta\sum_{k=0}^{n} r_{i,k} + (1-\beta)\sum_{k=0}^{n} l_{i,k}\right] - \left[\beta\sum_{k=0}^{n} r_{j,k} + (1-\beta)\sum_{k=0}^{n} l_{j,k}\right]\right\}. \end{aligned} \tag{5}$$

The above equation indicates that the total dominance is actually a comparison function. The larger the index of optimism β implies that the right dominance is more important. Herein, the index of optimism is used to reflect a decision maker's degree of optimism. A more optimistic decision maker generally takes a larger value of the index, for example, a situation in which $\beta = 1$ (or 0) represents an optimistic (pessimistic) decision maker's perspectives, and only right (left) dominance is considered. Furthermore, the total dominance of one fuzzy number over the other equals the difference between the two convex combinations using the respective left and right spreads. Based on the previous equation, the convex combination of a fuzzy number's left and right spreads is Yager's F_3 index, the total integral value with $\gamma = 0.5$, and Fortemps and Roubens' area compensation, when $\beta = 0.5$ and $n \to \infty$. Therefore, the value of $D_{i,j}$ (0.5) is the difference of the F_3 values and the total integral values with $\gamma = 0.5$ of A_i and A_j, and equals to Fortemps and Roubens' area compensation between A_i and A_j, when $n \to \infty$.

A decision maker can rank a pair of fuzzy numbers, A_i and A_j, using $D_{i,j}(\beta)$ based on the following rules:

(1) if $D_{i,j}(\beta) > 0$, then $A_i > A_j$;
(2) if $D_{i,j}(\beta) = 0$, then $A_i = A_j$; and
(3) if $D_{i,j}(\beta) < 0$, then $A_i < A_j$.

An illustrative example compares the two fuzzy numbers, while considering different degrees of optimism. In particular, two examples are used to illustrate the proposed approach's ability to rank discrete fuzzy numbers and the combination case of a crisp number and fuzzy numbers.

EXAMPLE 1. Figure 2 illustrates two triangular fuzzy numbers, $A_1 = [94/35, 46/7, 46/7, 10]$ and $A_2 = [2, 7, 7, 9]$, cited from [5]. By using equation (5) with $\beta = 0, 0.5$, and 1.0, and letting $n = 5$, the resulting total dominance of A_1 over A_2 is $D_{1,2}(0.0) = 0.129$, $D_{1,2}(0.5) = 0.207$, and $D_{1,2}(1.0) = 0.286$, respectively. Therefore, the same conclusion is reached, regardless of the value of β; i.e., $A_1 > A_2$. This conclusion correlates with that in [11].

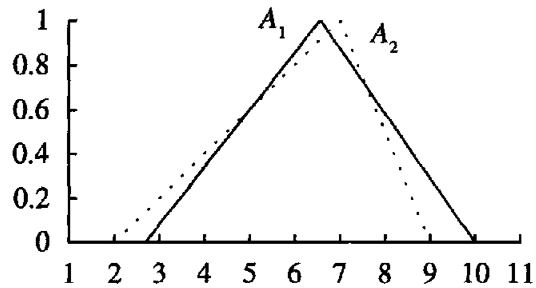

Figure 2. Fuzzy numbers $A_1 = [94/35, 46/7, 46/7, 10]$ and $A_2 = [2, 7, 7, 9]$.

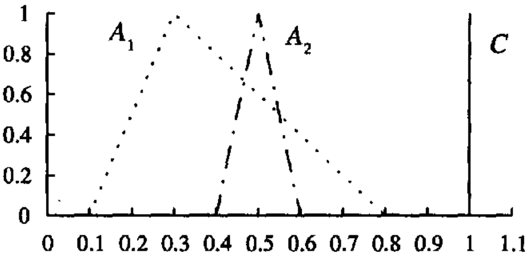

Figure 3. Fuzzy numbers $A_1 = [0.1, 0.3, 0.3, 0.8]$, $A_2 = [0.4, 0.5, 0.5, 0.6]$, and $C = 1$.

EXAMPLE 2. Two discrete fuzzy numbers, cited from [1], are defined as

$$\mu_{A_1} = \frac{1}{4} + \frac{0.75}{5} + \frac{0.5}{6} + \frac{0.25}{7}, \qquad \mu_{A_2} = \frac{0.5}{4} + \frac{1.0}{5} + \frac{0.5}{6}.$$

Likewise, the total dominance equation can effectively handle ordering discrete fuzzy numbers. In this case, the resulting total dominance of A_1 over A_2 with $\beta = 0.5$ and $n = 5$ if $D_{1,2}(0.5) = -0.3$, and $A_1 < A_2$ is concluded, which conforms to Chen and Hwang's outcome using Mabuchi's approach [21].

EXAMPLE 3. Consider a crisp number $C = 1$, and two fuzzy numbers, $A_1 = [0.1, 0.3, 0.3, 0.8]$ and $A_2 = [0.4, 0.5, 0.5, 0.6]$, also cited from [1], as shown in Figure 3. To obtain the left and right dominance, a crisp number is regarded as one fuzzy number having the same left and right spreads (equivalent to 1, in this example) at each α-level. Letting $\beta = 0.5$ and using six cuts ($n = 5$), the resulting total dominance is $D_{1,2}(0.5) = -0.125$ and $D_{2,c}(0.5) = -0.5$. It means that $A_1 < A_2$ and $A_2 < C$, such that $A_1 < A_2 < C$. This conclusion corresponds to human intuition as in [1]. Moreover, this kind of comparison cannot be achieved by the methods mentioned above, i.e., the Yager's F_3 index, the total integral value, and Fortemps and Roubens' area compensation.

3. SOME PROPERTIES

Based on the above definition, the total dominance of one fuzzy number over the other is the measurement of the degree of their difference based on the left and right spreads. The fact that the left and right spreads represent fuzzy numbers' relative locations on the X-axis accounts for why the value of the total dominance can be used to rank fuzzy numbers by natural ordering. Some valuable properties are described in the following, which are useful in ranking a large quantity of fuzzy numbers simultaneously. Assume that there are m different bounded fuzzy numbers, $A_1, A_2, \ldots, A_m$, to be ranked. Let A_i, A_j, and A_k be any three arbitrary fuzzy numbers, where $i \neq j \neq k$ and $1 \leq i, j, k \leq m$.

(1) The total dominance of a fuzzy number over itself is null; i.e.,

$$D_{i,i}(\beta) = 0, \qquad \text{for any } i \text{ and } \beta. \tag{6}$$

(2) The total dominance of A_i over A_j is opposite to that of A_j over A_i; i.e.,

$$D_{i,j}(\beta) = -D_{j,i}(\beta), \qquad \text{for any } i \text{ and } j, \text{ and } \beta. \tag{7}$$

(3) For A_i, A_j, and A_k, the transitivity property for the total dominance exists between them; i.e.,

$$\text{if } D_{i,j}(\beta) > 0 \quad \text{and} \quad D_{j,k}(\beta) > 0, \qquad \text{then } D_{i,k}(\beta) > 0. \tag{8}$$

Therefore, if $A_i > A_j$ and $A_j > A_k$ are known, we can infer that $A_i > A_k$. In fact, the total dominance among three fuzzy numbers has the following relation:

$$D_{i,k}(\beta) = D_{i,j}(\beta) + D_{j,k}(\beta). \tag{9}$$

Restated, once the values of $D_{i,j}(\beta)$ and $D_{j,k}(\beta)$ are known, the value of $D_{i,k}(\beta)$ is determined by simple arithmetic computations. Then, the order of A_i and A_k is obtained on the basis of the sign of $D_{i,k}(\beta)$.

(4) More than two fuzzy numbers can be ranked by comparing with the benchmark fuzzy number. Let A_j be the benchmark, and $D_{i,j}(\beta) = a$ and $D_{k,j}(\beta) = b$. By using the previous two properties, obviously $D_{i,k}(\beta) = D_{i,j}(\beta) - D_{k,j}(\beta) = a - b$. Therefore, if $a > b$, then $D_{i,k}(\beta) > 0$; i.e., $A_i > A_k$.

(5) The ranking of more than two fuzzy numbers has the robustness property [27]; i.e.,

$$\text{if } D_{i,j}(\beta) < \varepsilon, \qquad \text{then } |D_{i,k}(\beta) - D_{j,k}(\beta)| < \varepsilon. \tag{10}$$

This equation suggests that the total dominance difference between one fuzzy number and the other two fuzzy numbers is insignificant, if the two fuzzy numbers are close to each other. This property holds since $D_{i,k}(\beta) - D_{j,k}(\beta) = D_{i,j}(\beta)$.

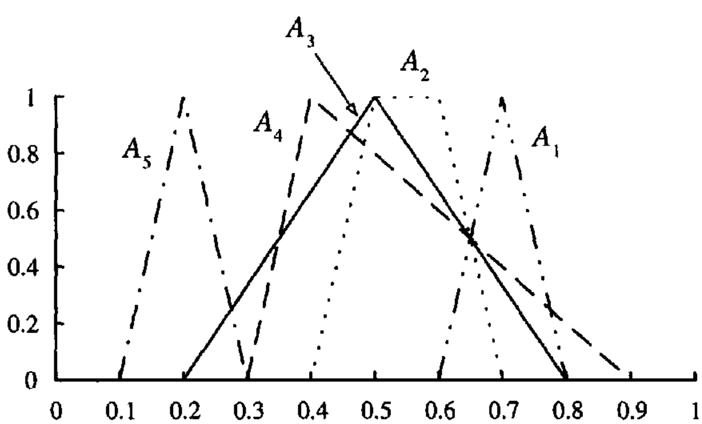

Figure 4. Five fuzzy numbers in Example 4.

Table 1. The total dominance using A_3 as the benchmark fuzzy number.

	$\beta = 0.0$	$\beta = 0.5$	$\beta = 1.0$
$D_{1,3}(\beta)$	0.3	0.2	0.1
$D_{2,3}(\beta)$	0.1	0.05	0.0
$D_{4,3}(\beta)$	0.0	0.0	0.0
$D_{5,3}(\beta)$	−0.2	−0.3	−0.4

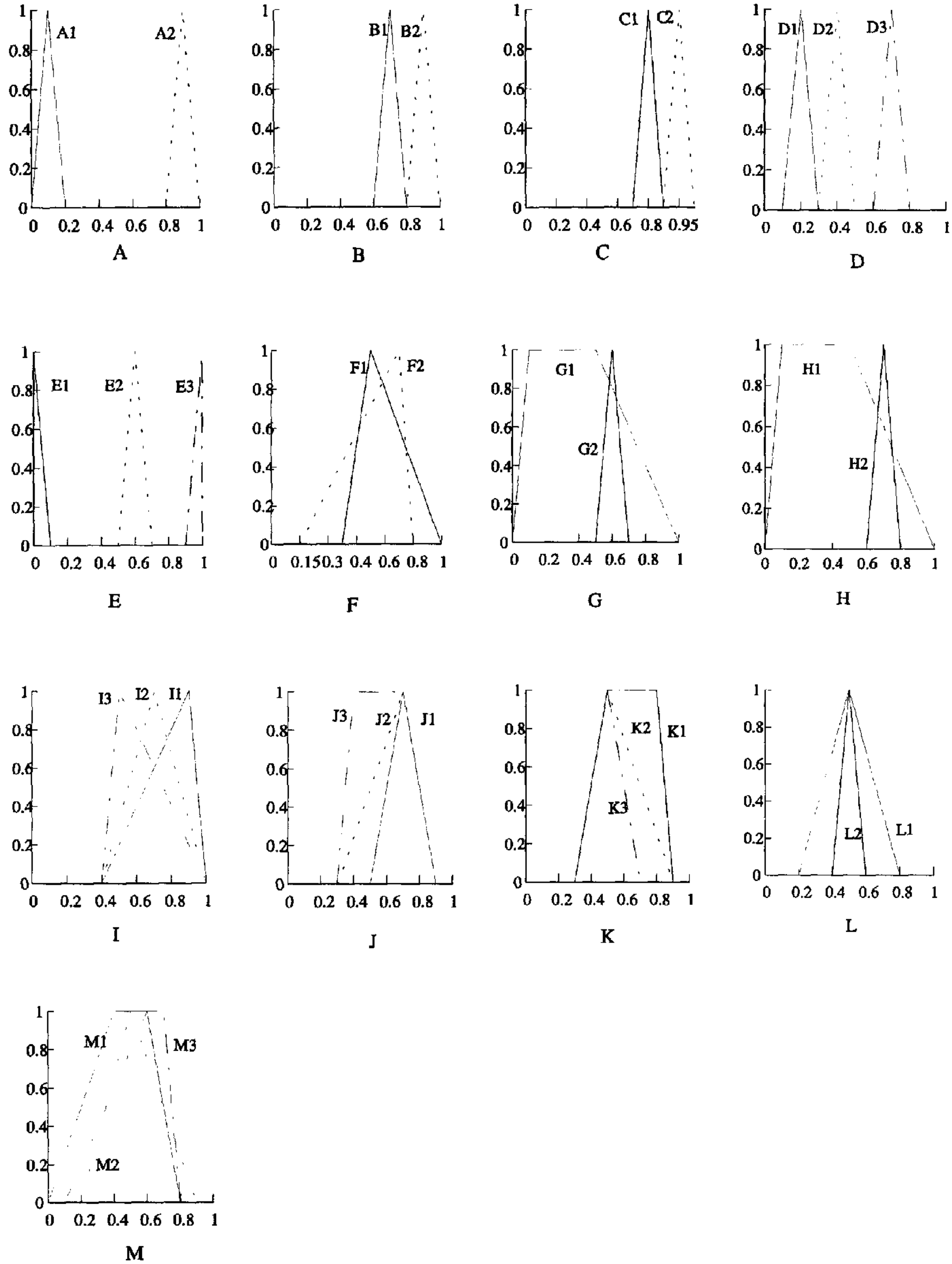

Figure 5. Examples given by Bortolan and Degani.

For ranking m fuzzy numbers, only $m-1$ comparisons to the benchmark fuzzy number are necessary when using the above properties, instead of $m(m-1)/2$ or m comparisons in the related articles [23,29]. For example, letting A_j be the benchmark, then only $m-1$ values of the total dominance, $D_{1,j}(\beta), D_{2,j}(\beta), \dots, D_{j-1,j}(\beta), D_{j+1,j}(\beta), \dots, D_{m,j}(\beta)$, are necessarily determined.

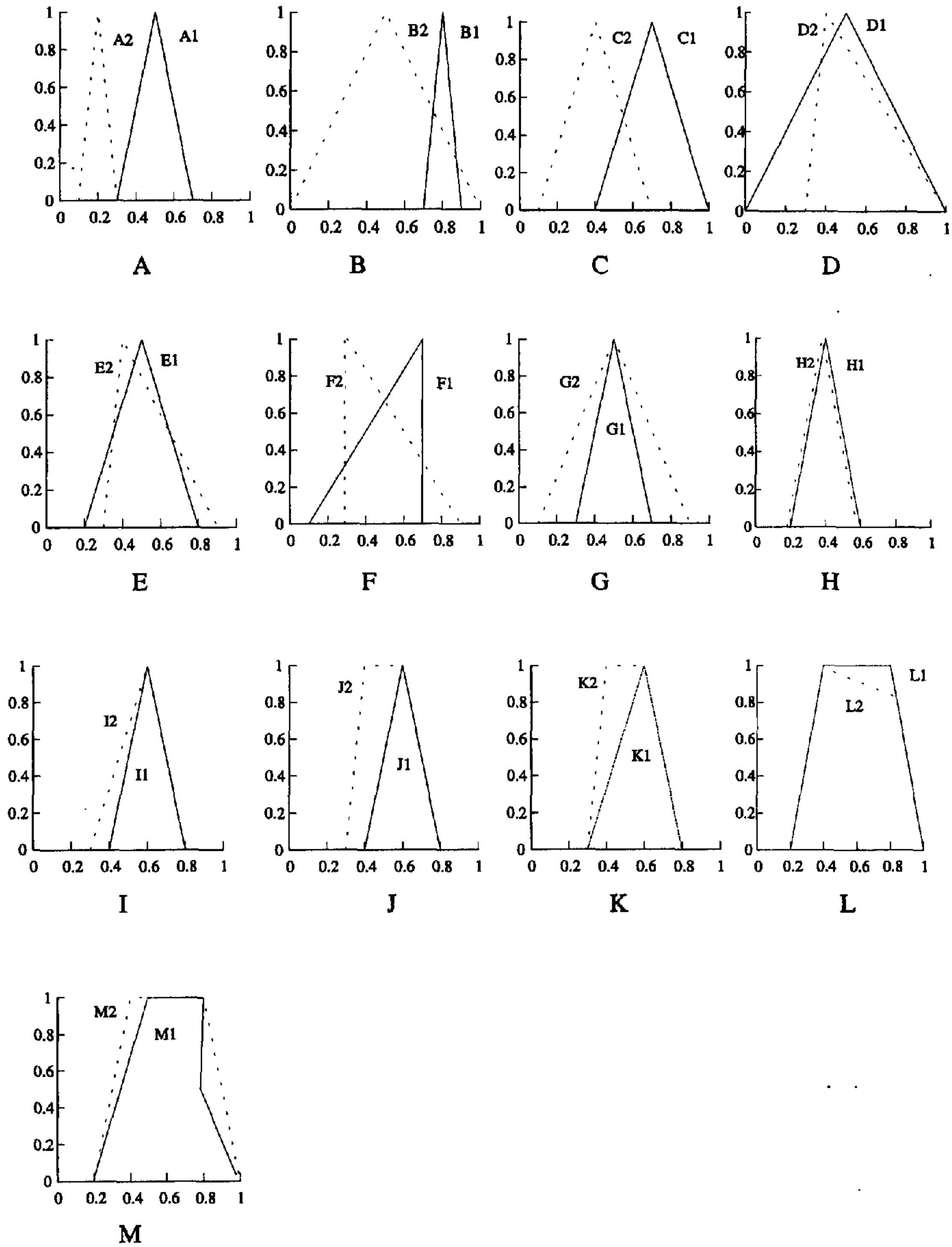

Figure 6. Examples given by Tseng and Klein.

Once these values are known, natural orderings easily determine the rankings. This approach is more efficient than the existing ranking methods. The following example, as adopted from [23], is used to demonstrate the proposed approach's efficiency.

EXAMPLE 4. Assume that five fuzzy numbers are to be ranked. They are defined as $A_1 = [0.6, 0.7, 0.7, 0.8]$, $A_2 = [0.4, 0.5, 0.6, 0.7]$, $A_3 = [0.2, 0.5, 0.5, 0.8]$, $A_4 = [0.3, 0.4, 0.4, 0.9]$, $A_5 =$

Table 2. The values of the total dominance in Bortolan and Degani's examples based on the index of optimism $\beta = 0.5$ and six different numbers of cuts.

Examples	$n = 2$	$n = 3$	$n = 5$	$n = 7$	$n = 10$	$n = 20$
A	−0.8	−0.8	−0.8	−0.8	−0.8	−0.8
B	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2
C	−0.15	−0.15	−0.15	−0.15	−0.15	−0.15
D	−0.2 −0.3	−0.2 −0.3	−0.2 −0.3	−0.2 −0.3	−0.2 −0.3	−0.2 −0.3
E	−0.575 −0.375	−0.575 −0.375	−0.575 −0.375	−0.575 −0.375	−0.575 −0.375	−0.575 −0.375
F	0.0	0.0	0.0	0.0	0.0	0.0
G	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2
H	−0.3	−0.3	−0.3	−0.3	−0.3	−0.3
I	0.1 0.1	0.1 0.1	0.1 0.1	0.1 0.1	0.1 0.1	0.1 0.1
J	0.05 0.075	0.05 0.075	0.05 0.075	0.05 0.075	0.05 0.075	0.05 0.075
K	0.075 0.05	0.075 0.05	0.075 0.05	0.075 0.05	0.075 0.05	0.075 0.05
L	0.0	0.0	0.0	0.0	0.0	0.0
M	−0.075 −0.025	−0.075 −0.025	−0.075 −0.025	−0.075 −0.025	−0.075 −0.025	−0.075 −0.025

Note: (1) For examples with two fuzzy numbers, the value in the cell is $D_{1,2}(0.5)$. (2) For examples with three fuzzy numbers, the values in the cell are $D_{1,2}(0.5)$ and $D_{2,3}(0.5)$.

Table 3. The values of the total dominance in Tseng and Klein's examples based on the index of optimism $\beta = 0.5$ and six different numbers of cuts.

	$n = 2$	$n = 3$	$n = 5$	$n = 7$	$n = 10$	$n = 20$
A	0.3	0.3	0.3	0.3	0.3	0.3
B	0.3	0.3	0.3	0.3	0.3	0.3
C	0.3	0.3	0.3	0.3	0.3	0.3
D	−0.025	−0.025	−0.025	−0.025	−0.025	−0.025
E	0.0	0.0	0.0	0.0	0.0	0.0
F	0.1	0.1	0.1	0.1	0.1	0.1
G	0.0	0.0	0.0	0.0	0.0	0.0
H	0.05	0.05	0.05	0.05	0.05	0.05
I	0.025	0.025	0.025	0.025	0.025	0.025
J	0.075	0.075	0.075	0.075	0.075	0.075
K	0.05	0.05	0.05	0.05	0.05	0.05
L	0.067	0.050	0.035	0.033	0.029	0.025
M	0.008	0.0	0.005	0.004	0.002	0.0

Note: The value in the cell is $D_{1,2}(0.5)$.

[0.1, 0.2, 0.2, 0.3]. Figure 4 depicts their membership functions. Also, assume that A_3 is selected as the benchmark in this case. With six α-cuts (i.e., $n = 5$), the total dominance of each fuzzy number over A_3 is obtained based on different indices of optimism. Table 1 summarizes those results. According to this table, the ranking orders are

(1) $A_1 > A_2 > A_3 = A_4 > A_5$, when $\beta = 0.0$;
(2) $A_1 > A_2 > A_3 = A_4 > A_5$, when $\beta = 0.5$; and
(3) $A_1 > A_2 = A_3 = A_4 > A_5$, when $\beta = 1$.

The orders are the same when $\beta = 0.0$ or 0.5, while A_2 is equal to A_3 and A_4 if the decision maker is optimistic. Notably, this example only requires four comparisons.

Table 4. Cases studies using Bortolan and Degani's examples.

	A1	A2	B1	B2	C1	C2	D1	D2	D3	E1	E2	E3	F1	F2	G1	G2
Yager F1	0.10	0.90	0.70	0.90	0.80	0.95	0.20	0.40	0.70	0.03	0.60	0.97	0.61	0.53	0.41	0.60
F2	0.18	0.90	0.72	0.90	0.81	0.95	0.27	0.45	0.72	0.09	0.63	1	0.66	0.69	0.66	0.63
F3	0.10	0.90	0.70	0.90	0.80	0.95	0.20	0.40	0.70	0.02	0.60	0.97	0.58	0.56	0.40	0.60
Chang	0.02	0.18	0.14	0.18	0.16	0.09	0.04	0.08	0.14	0	0.12	0.10	0.40	0.34	0.58	0.12
Adamo 0.9M	0.11	0.91	0.71	0.91	0.81	0.95	0.21	0.41	0.71	0.01	0.61	1	0.55	0.56	0.55	0.61
0.9m	0.11	0.91	0.71	0.91	0.81	0.95	0.21	0.41	0.71	0.01	0.61	1	0.55	0.66	0.55	0.61
0.5	0.15	0.95	0.75	0.95	0.85	0.97	0.25	0.45	0.75	0.05	0.65	1	0.75	0.72	0.75	0.65
Baas-Kwakernaak	0	1	0	1	0	1	0	0	1	0	0	1	0.84	1	0.82	1
Baldwin-Guild l.p.	0	0.82	0	0.32	0	0.25	0	0	0.40	0	0	0.45	0.42	0.33	0.30	0.58
g.	0	0.82	0	0.47	0	0.40	0	0	0.44	0	0	0.67	0.44	0.37	0.36	0.42
r.a.	0	0.69	0	0.20	0	0.14	0	0	0.30	0	0	0.27	0.34	0.24	0.21	0.55
Kerre	0.80	1	0.80	1	0.85	1	0.80	0.80	1	0.89	0.85	1	0.96	0.89	0.51	0.89
Jain $K = 1$	0.18	0.90	0.72	0.90	0.81	0.95	0.32	0.55	0.89	0.09	0.63	1	0.66	0.69	0.66	0.63
$K = 2$	0.03	0.84	0.55	0.84	0.70	0.92	0.12	0.33	0.80	0	0.42	1	0.53	0.51	0.53	0.42
$K = 1/2$	0.40	0.95	0.84	0.95	0.90	0.97	0.55	0.72	0.94	0.26	0.78	1	0.78	0.81	0.78	0.78
Dubois-Prade PD	0	1	0	1	0	1	0	0	1	0	0	1	0.84	1	0.82	1
PSD	0	1	0	1	0	1	0	0	1	0	0	1	0.54	0.46	0.66	0.32
ND	0	1	0	1	0	1	0	0	1	0	0	1	0.54	0.46	0	1
NSD	0	1	0	1	0	1	0	0	1	0	0	1	0	0.16	0	0.18
Kim-Park $K = 1$	0.18	0.91	0.40	0.80	0.50	0.86	0.25	0.50	0.88	0.09	0.64	1	0.63	0.65	0.67	0.64
$K = 0.5$	0.14	0.86	0.30	0.70	0.38	0.79	0.19	0.44	0.81	0.05	0.59	0.95	0.49	0.51	0.38	0.59
$K = 0$	0.09	0.82	0.20	0.60	0.25	0.71	0.13	0.34	0.75	0	0.55	0.91	0.35	0.37	0.09	0.55
Fortemps-Roubens	0.10	0.90	0.70	0.90	0.80	0.95	0.20	0.40	0.70	0.03	0.60	0.98	0.58	0.53	0.40	0.60
Lee-Lee O	0	1	0	1	0	1	0	0	1	0	0	1	0.67	0.32	0.40	0.60
G	0	1	0	1	0	1	0	0.50	1	0	0.50	1	1	0.32	0.23	1
Liou-Wang $\gamma = 1$	0.15	0.95	0.75	0.95	0.85	0.98	0.25	0.45	0.75	0.5	0.65	1	0.75	0.75	0.75	0.65
$\gamma = 0.5$	0.1	0.9	0.7	0.9	0.8	0.95	0.2	0.4	0.7	0.25	0.6	0.98	0.59	0.59	0.4	0.6
$\gamma = 0$	0.05	0.85	0.65	0.85	0.75	0.93	0.15	0.35	0.65	0	0.55	0.95	0.43	0.43	0.05	0.55
Chen-Lu $\beta = 1$	−0.800		−0.200		−0.125		−0.200		−0.300	−0.150		−0.350	0.000		0.100	
$\beta = 0.5$	−0.800		−0.200		−0.150		−0.200		−0.300	−0.350		−0.375	0.000		−0.200	
$\beta = 0$	−0.800		−0.200		−0.175		−0.200		−0.300	−0.550		−0.400	0.000		−0.500	

Note: The values of the total dominance (Chen-Lu) in the table are $D_{1,2}(\beta)$ (and $D_{2,3}(\beta)$).

Table 4. (continued).

	H1	H2	I1	I2	I3	J1	J2	J3	K1	K2	K3	L1	L2	M1	M2	M3
Yager F1	0.41	0.70	0.76	0.7	0.63	0.70	0.63	0.57	0.62	0.56	0.50	0.50	0.50	0.44	0.53	0.52
F2	0.66	0.72	0.90	0.76	0.66	0.75	0.75	0.75	0.62	0.64	0.58	0.61	0.50	0.66	0.64	0.72
F3	0.40	0.70	0.80	0.70	0.60	0.70	0.65	0.57	0.62	0.54	0.50	0.50	0.50	0.45	0.52	0.55
Chang	0.58	0.14	0.46	0.41	0.38	0.28	0.37	0.52	0.56	0.33	0.20	0.29	0.10	0.43	0.37	0.42
Adamo 0.9M	0.55	0.71	0.91	0.73	0.55	0.72	0.72	0.72	0.81	0.54	0.52	0.53	0.51	0.62	0.54	0.71
0.9m	0.55	0.71	0.91	0.73	0.55	0.72	0.72	0.72	0.81	0.54	0.52	0.53	0.51	0.62	0.54	0.71
0.5	0.75	0.75	0.95	0.85	0.75	0.80	0.80	0.80	0.85	0.70	0.60	0.65	0.55	0.70	0.70	0.75
Baas-Kwakernaak	0.66	1	1	0.74	0.60	1	1	1	1	1	1	1	1	1	0.88	1
Baldwin-Guild l.p.	0.24	0.66	0.42	0.33	0.30	0.37	0.27	0.27	0.45	0.37	0.27	0.27	0.28	0.40	0.42	0.42
g.	0.30	0.54	0.55	0.40	0.34	0.42	0.35	0.35	0.53	0.40	0.28	0.30	0.24	0.40	0.42	0.44
r.a.	0.16	0.60	0.28	0.23	0.22	0.27	0.19	0.19	0.31	0.28	0.21	0.20	0.23	0.30	0.34	0.32
Kerre	0.42	0.95	1	0.86	0.76	1	0.91	0.75	1	0.85	0.75	0.91	0.91	0.76	0.92	0.96
Jain $K = 1$	0.66	0.72	0.90	0.76	0.66	0.82	0.82	0.82	0.90	0.69	0.64	0.73	0.67	0.73	0.69	0.80
$K = 2$	0.53	0.54	0.84	0.65	0.54	0.71	0.71	0.71	0.82	0.56	0.45	0.60	0.48	0.58	0.56	0.67
$K = 1/2$	0.78	0.84	0.95	0.86	0.78	0.89	0.89	0.89	0.94	0.80	0.77	0.83	0.80	0.84	0.80	0.89
Dubois-Prade PD	0.66	1	1	0.74	0.60	1	1	1	1	1	1	1	1	1	0.88	1
PSD	0.50	0.50	0.74	0.23	0.16	0.50	0.50	0.50	0.80	0.20	0	0.73	0.24	0.30	0.40	0.60
ND	0	1	0.63	0.38	0.18	0.67	0.35	0	0.50	0.50	0.50	0.27	0.76	0.30	0.50	0.50
NSD	0	0.34	0.26	0	0	0	0	0	0	0	0	0	0	0	0	0
Kim-Park $K = 1$	0.67	0.55	0.86	0.67	0.55	0.75	0.75	0.75	0.78	0.58	0.44	0.67	0.57	0.73	0.69	0.80
$K = 0.5$	0.38	0.68	0.66	0.50	0.34	0.63	0.58	0.45	0.50	0.40	0.33	0.50	0.50	0.52	0.55	0.61
$K = 0$	0.09	0.64	0.45	0.33	0.14	0.50	0.40	0.14	0.22	0.22	0.22	0.33	0.43	0.31	0.42	0.43
Fortemps-Roubens	0.40	0.70	0.80	0.70	0.60	0.70	0.65	0.58	0.63	0.55	0.50	0.50	0.50	0.45	0.53	0.55
Lee-Lee O	0.40	0.70	0.65	0.35	0.20	0.65	0.34	0.24	0.62	0.38	0.24	0.50	0.50	0.35	0.50	0.55
G	0.13	1	1	0.61	0.18	1	0.60	0.23	1	0.62	0.21	1	1	0.41	1	1
Liou-Wang $\gamma = 1$	0.75	0.75	0.95	0.85	0.75	0.8	0.8	0.8	0.85	0.7	0.6	0.65	0.55	0.7	0.7	0.75
$\gamma = 0.5$	0.4	0.7	0.8	0.7	0.6	0.7	0.65	0.58	0.63	0.55	0.5	0.5	0.5	0.45	0.53	0.55
$\gamma = 0$	0.05	0.65	0.65	0.55	0.45	0.6	0.5	0.35	0.4	0.4	0.4	0.35	0.45	0.2	0.35	0.35
Chen-Lu $\beta = 1$	0.000		0.100		0.100	0.000		0.000	0.150		0.100	0.100		0.000		0.050
$\beta = 0.5$	−0.300		0.100		0.100	0.050		0.075	0.075		0.050	0.000		−0.075		−0.025
$\beta = 0$	−0.600		0.100		0.100	0.100		0.150	0.000		0.000	−0.100		−0.150		0.000

4. COMPARATIVE EXAMPLES

The applicability of the proposed approach is demonstrated by comparing the ranking orders with those of some existing ranking methods using two groups of examples. Each group has thirteen examples. One group, as shown in Figure 5, is adopted from [28], while the other is adopted from [23] and is illustrated in Figure 6. For comparison, two parameters, i.e., the index of optimum and the number of α-cuts, should be determined before calculating the total dominance of one fuzzy number over the other based on equation (5). First, six different numbers of α-cuts ($n = 2, 3, 5, 7, 10, 20$) and $\beta = 0.5$ are used to investigate how the number of α-cuts influences the total dominance of examples in each group. Tables 2 and 3 list the outcomes of examples in Figures 5 and 6, respectively. Closely examining the two tables reveals that the ranking orders

Table 5. Cases studies using Tseng and Klein's examples.

Part I.

	A1	A2	B1	B2	C1	C2	D1	D2	E1	E2	F1	F2	G1	G2
Tseng and Klein	1.0	0.0	0.87	0.13	0.87	0.13	0.47	0.53	0.49	0.51	0.56	0.44	0.50	0.50
Kolodziejczyk's R1	1.0	0.0	0.87	0.13	0.87	0.13	0.47	0.53	0.49	0.51	0.56	0.44	0.50	0.50
Kerre	1.0	0.70	0.99	0.54	1.0	0.55	0.89	0.95	0.95	0.96	0.93	0.87	0.90	0.90
Baldwin-Guild	0.46	0.0	0.56	0.19	0.56	0.19	0.44	0.48	0.36	0.39	0.40	0.36	0.38	0.38
Chen-Lu $\beta = 1$	0.350		0.100		0.300		0.050		0.00		0.100		−0.100	
$\beta = 0.5$	0.300		0.300		0.300		−0.030		0.00		0.100		0.000	
$\beta = 0$	0.250		0.500		0.300		−0.100		0.00		0.100		0.100	

Note: The values of the total dominance (Chen-Lu) in the table are $D_{1,2}(\beta)$.

Part II.

	H1	H2	I1	I2	J1	J2	K1	K2	L1	L2	M1	M2
Tseng and Klein	0.52	0.48	0.56	0.44	0.64	0.36	0.58	0.42	0.52	0.48	0.50	0.50
Kolodziejczyk's R1	0.52	0.48	0.56	0.44	0.64	0.36	0.58	0.42	0.52	0.48	0.50	0.50
Kerre	1.0	0.98	1.0	0.95	1.0	0.85	1.0	0.90	1.0	0.96	0.95	0.95
Baldwin-Guild	0.29	0.28	0.33	0.29	0.38	0.29	0.38	0.33	0.57	0.44	0.57	0.53
Chen-Lu $\beta = 1$	0.020		0.000		0.000		0.000		0.070		−0.040	
$\beta = 0.5$	0.020		0.025		0.075		0.050		0.050		0.005	
$\beta = 0$	0.020		0.050		0.150		0.100		0.000		0.050	

Note: The values of the total dominance (Chen-Lu) in the table are $D_{1,2}(\beta)$.

are consistent regardless of the number of α-cuts, except for a particular Example M in Table 3. This investigation illustrates that only a smaller number of α-cuts is necessary, if the membership functions of fuzzy numbers are simple (such as triangular or trapezoidal), as most examples in the two groups. In addition, the total dominance between the two fuzzy numbers in Example M of Table 3 is insignificant, and they equal each other when the number of α-cuts increases ($n = 20$). According to the definition of the total dominance in this study, the use of a greater number of α-cuts can obviously produce a more accurate ordering. Therefore, we can conclude that the two fuzzy numbers, M_1 and M_2, in Table 3 equal each other, which corresponds to [23]. In fact, according to the definition of fuzzy sets [2,31,32], the ranking of two fuzzy numbers based on a minute difference is not very meaningful.

In light of the outcomes in Tables 2 and 3, a great number of α-cuts appear to be unnecessary for obtaining an accurate ordering since the total dominance values are consistent in most examples at any number of α-cuts. Therefore, we use six cuts ($n = 5$) and three types of indices of optimism ($\beta = 0.0, 0.5$, and 1.0) to compare the ranking orders with those produced by the existing ranking methods using the two groups of examples. Related to Figure 5, Table 4 lists the results reproduced from [9,27,28] and the results by the total integral values with three kinds of γ values. In the table, the area compensation of two fuzzy numbers using Fortemps and Roubens' method is the difference of the corresponding two values in the example. The outcomes using the left and right dominance based on $n = 5$ and three kinds of β values are shown in the last row. Previous works [3,9,26–28] have indicated that some ranking methods fail to provide ranking orders even for simple examples. For example, Bass-Kwakernaak [17], Kerre [8], Baldwin-Guild [16], and Dubois and Prade [20] cannot discriminate fuzzy numbers in Examples D and E. For Example J, some ranking methods [4,6,7,13,16,17,20] have a deficiency in comparison.

For most examples in Table 4, the total dominance values with $\beta = 0.5$ ($D_{1,2}(0.5)$ and $D_{2,3}(0.5)$) obviously correspond to the differences of the values of the two comparison indices using Yager's F_3, the total integral values with $\gamma = 0.5$, and Fortemps and Roubens' area compensation, as mentioned earlier. The ranking orders of some examples are changed when a decision maker's optimistic or pessimistic perspective is changed. Examples G, H, J, K, L, and M have different orderings if the index of optimism has different values. However, these examples correlate with those provided by Kim and Park using the same value of the index of optimism. As an exception, the two fuzzy numbers in Example F are determined to be equivalent based on the resulting total dominance at the three β values. This conclusion is not exactly the same as that given by Yager's F_3 [15] and Fortemps and Roubens [27], although their differences of the values of the two comparison indices in both methods are minimal.

Regarding the comparisons using Tseng and Klein's examples, the ranking orders based on the total dominance values at $\beta = 0.5$ correspond to those by the other methods, as listed in Table 5, with two exceptions, Examples E and M. However, the difference is negligible. Similar to the previous comparisons, the use of different indices of optimism could result in different orderings for some examples.

5. CONCLUSION

Ranking fuzzy numbers is a critical task in a fuzzy decision making process. Particularly, when ranking a large quantity of fuzzy numbers and only limited information about them can be obtained, an effective, efficient, and accurate ranking method becomes necessary. The proposed ranking approach only considers the left and right spreads at each α-level of fuzzy numbers to be ordered. The left and right spreads are used to determine the respective dominance through simple computations. Then, the total dominance is obtained depending on the decision maker's optimistic and pessimistic perspectives. The total dominance is actually the measurement of the degree of difference between two fuzzy numbers' related locations on the X-axis. A few properties described in the previous section are very useful for ranking a large quantity of fuzzy numbers. Based on the properties, only $m - 1$ total dominance values (pair-comparisons) are necessary for ranking m different fuzzy numbers. This makes the ranking process more efficient.

This paper does not compare nonnormal fuzzy numbers since the significance of this kind of comparison is unclear, and therefore, is quite debatable, as claimed by Bortolan and Degani [28]. Previous literature rarely addresses the feasibility of comparing two nonnormal fuzzy numbers. The proposed ranking approach can allow for the neglect of membership functions of fuzzy numbers to be ranked because of the only considerations of the left and right spreads. This is helpful in the decision making process when only limited information regarding fuzzy numbers can be acquired. The use of the total dominance has the merits of simple computations and fewer pair-comparisons, resulting in significant savings in time and in effort when ordering a large quantity of fuzzy numbers. Comparing the proposed approach with some existing ranking methods by two groups of examples reveals that the former uses only six cuts and, in doing so, yields consistent and accurate outcomes. This is despite the fact that few differences exist among the examples with the two fuzzy numbers that closely resemble each other or are close to each other. The proposed ranking approach also demonstrates that only a smaller number of α-cuts is necessary for obtaining an accurate ranking order, if the membership functions of fuzzy numbers are simple (such as triangular or trapezoidal).

In general, the proposed ranking approach is efficient and effective owing to the simplicity in only requiring a few left and right spreads and in computational efforts, transitivity in ranking a large quantity of fuzzy numbers, flexibility in allowing a decision maker's optimistic perspectives, and the ease of interpretation over the existing methods. These merits make the proposed approach highly promising for future applications.

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